PHYS 232 Jan. 25, 2025

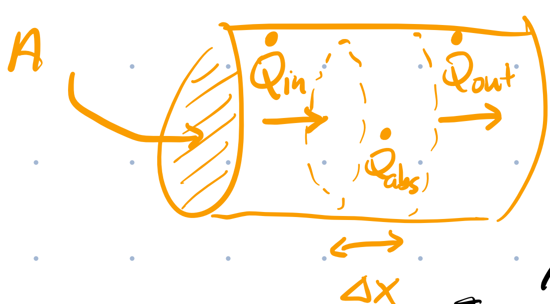
Alternative derivation of the heat equation and the key results for the thermal waves lab.

In the thermal waves lab, a heater is inserted into the end of a solid copper rod. The heat generated flows from the hot end by the heater towards the cool end. Part of the heat is absorbed by the copper causing it's temperature to rise according to :

absorbed heat

$$\Delta T = \frac{Q_{abs}}{c_v A \Delta x}$$

Where c_v is the specific heat per unit volume & $A\Delta x$ is the volume of copper that absorbed heat Q_{abs} & had a temp. change of ΔT.



Conservation of energy requires

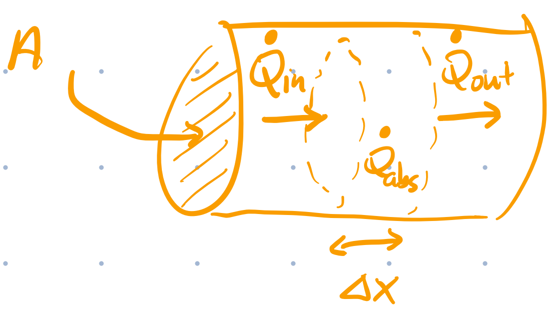
$$\dot{Q}_{in} = \dot{Q}_{abs} + \dot{Q}_{out}$$

Thermal version of Kirchoff's jcn rule.

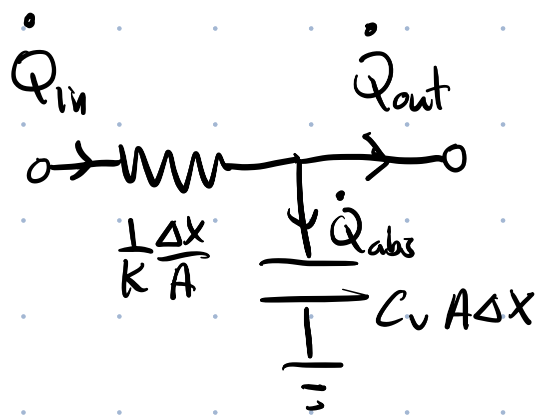
This scenario can be modeled as a "thermal circuit".

Here are the analogues:

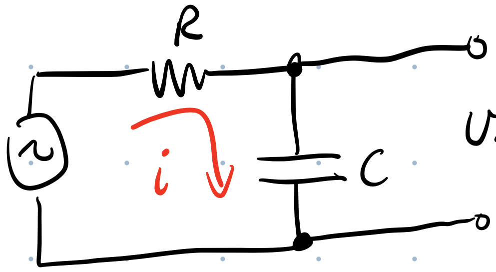
Electrical	Thermal
current I	heat flow rate $\dot{Q}$
voltage diff. ΔV	temp. diff. ΔT
capacitance C	heat capacity $C_v A \Delta x$
Resistance $R = \rho \frac{\Delta x}{A} = \frac{1}{\sigma} \frac{\Delta x}{A}$ (σ: electrical conductivity)	Thermal resistance $\frac{1}{K} \frac{\Delta x}{A}$ (K: thermal conductivity)



$\Longrightarrow$



Before moving on, we can remind ourselves of the behaviour of these kinds of RC circuits.



$v_{in} = V_i e^{j\omega t}$

$v_{out} = V_o e^{j(\omega t + \phi)}$

$$\frac{v_{out}}{v_{in}} = \frac{Z_c}{R + Z_c} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

$$= \frac{1}{1 + j\omega RC}\left(\frac{1 - j\omega RC}{1 - j\omega RC}\right) = \frac{1 - j\omega RC}{1 + (\omega RC)^2} \quad (1)$$

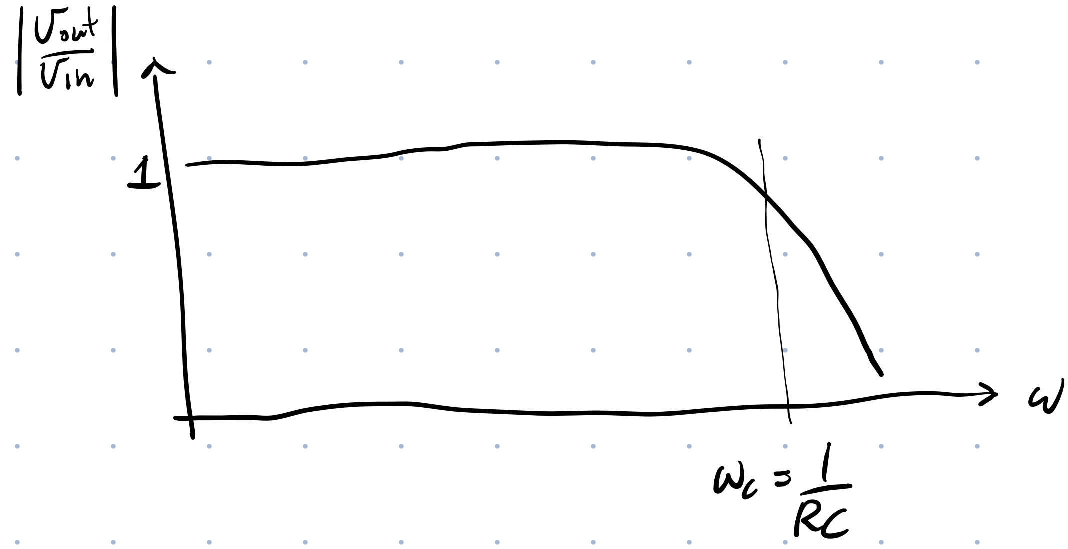
$$\therefore \left|\frac{v_{out}}{v_{in}}\right|^2 = \left[\frac{1 - j\omega RC}{1 + (\omega RC)^2}\right]\left[\frac{1 + j\omega RC}{1 + (\omega RC)^2}\right]$$

$$= \frac{1 + (\omega RC)^2}{\left[1 + (\omega RC)^2\right]^2} = \frac{1}{1 + (\omega RC)^2}$$

$$\boxed{\therefore \left|\frac{v_{out}}{v_{in}}\right| = \frac{1}{\sqrt{1 + (\omega RC)^2}}}$$

Low-Pass filter

v_{out} is attenuated relative to v_{in} when $\omega RC > 1$

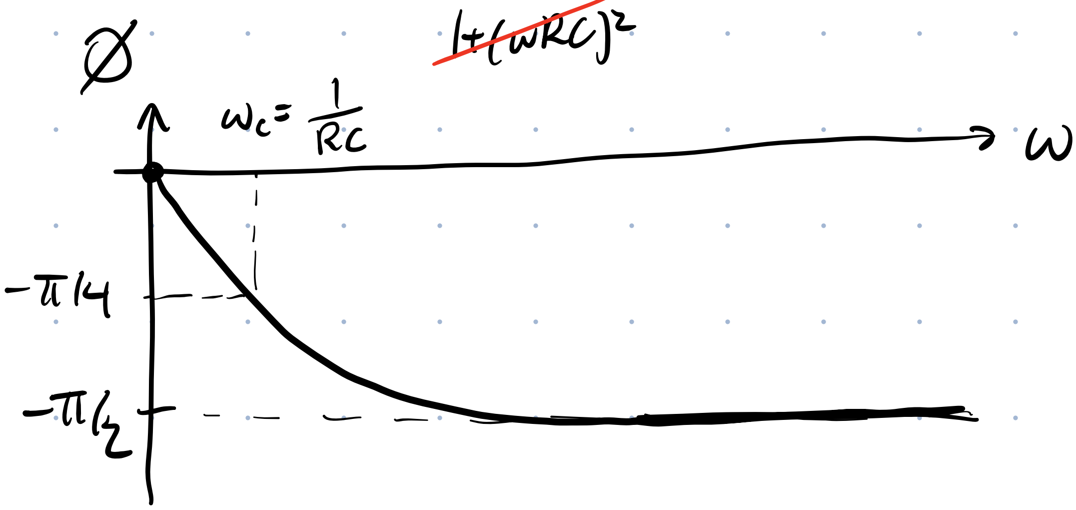


"cutoff" or "corner" frequency.

Returning to ①

$$\frac{V_{out}}{V_{in}} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$$

$$\tan\phi = \frac{\dfrac{-\omega RC}{\cancel{1+(\omega RC)^2}}}{\dfrac{1}{\cancel{1+(\omega RC)^2}}} = -\omega RC$$



$\therefore$ Not only does this RC circuit attenuate signals at high frequencies, it also introduces a phase shift that approaches $-\frac{\pi}{2}$ as frequency increases.

We should expect to see these same types of behaviour in our thermal circuit.

For our thermal circuit:

$$R_{Th} = \frac{1}{k}\frac{\Delta x}{A} \Rightarrow \frac{R_{Th}}{\Delta x} = \frac{1}{kA}$$

thermal resistance

$$C_{Th} = C_v A \Delta x \Rightarrow \frac{C_{Th}}{\Delta x} = C_v A$$

thermal capacitance

$$\therefore \frac{R_{Th} C_{Th}}{(\Delta x)^2} = \frac{C_v}{k} \equiv \frac{1}{\alpha}$$ where α is called the thermal diffusivity

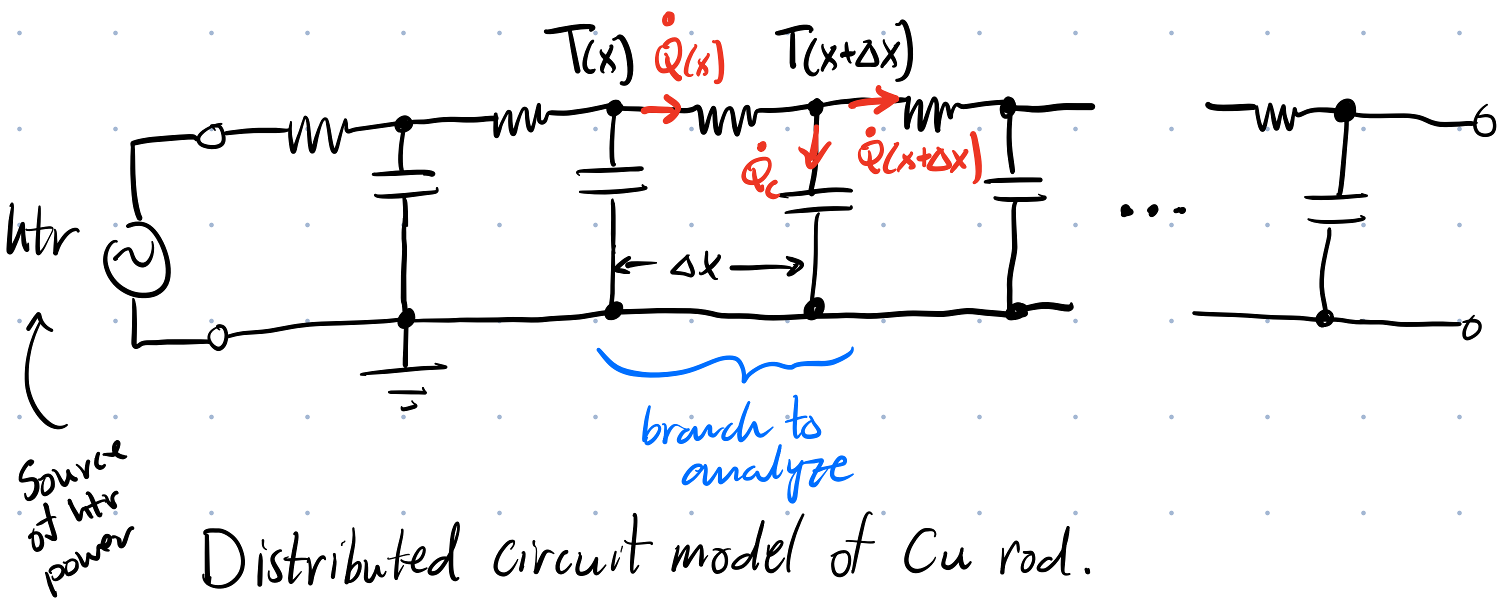
Near room temp: $k \simeq 400 \frac{W}{m \cdot K}$ $\quad C_v \simeq 3.4 \times 10^5 \frac{J}{m^3 \cdot K}$

$$\therefore \alpha = \frac{K}{C_V} \approx 1.2\times10^{-3} \frac{W}{m\cdot K} \frac{m^3 K}{J}$$

$$\boxed{\alpha = 1.2\times10^{-3} \frac{m^2}{s}}$$

The goal of the thermal waves exp4 is to meas. α.

We can model a long copper rod w/ a heater in one end, not as a single RC circuit, but as a daisy-chain of many cascaded RC branches:



Distributed circuit model of Cu rod.

This is a type of transmission line, a topic we'll discuss in some detail in the lectures of PHYS 331.

To make progress, we start by analyzing a single branch.

$$T(x) - \dot{Q}(x) \underbrace{\frac{1}{K}\frac{\Delta x}{A}}_{R_{th}} = T(x+\Delta x) \qquad (1a)$$

$$\dot{Q}(x) = \dot{Q}_c + \dot{Q}(x+\Delta x)$$

$$C_v A \Delta x = \frac{Q_c}{T(x+\Delta x)} \Rightarrow \dot{Q}_c = C_v A \Delta x \, \dot{T}(x+\Delta x)$$

$$\therefore \dot{Q}(x) = C_v A \Delta x \, \dot{T}(x+\Delta x) + \dot{Q}(x+\Delta x) \qquad (1b)$$

Now, rearrange (1a) & (2a)

$$\frac{T(x+\Delta x) - T(x)}{\Delta x} = -\frac{1}{KA}\dot{Q}(x)$$

$$\frac{\dot{Q}(x+\Delta x) - \dot{Q}(x)}{\Delta x} = -C_v A \, \dot{T}(x+\Delta x)$$

Clearly our analysis is an approximation. However, it becomes exact in the limit $\Delta x \to 0$.

Coupled partial differential eq'ns.

$$\begin{cases} \dfrac{\partial T(x,t)}{\partial x} = -\dfrac{1}{KA}\dot{Q}(x,t) & \text{(2a)} \\[2ex] \dfrac{\partial \dot{Q}(x,t)}{\partial x} = \boxed{-C_v A \dfrac{\partial T(x,t)}{\partial t}} & \text{(2b)} \end{cases}$$

sub

Take a second x-derivative of (2a):

$$\frac{\partial^2 T(x,t)}{\partial x^2} = -\frac{1}{KA}\boxed{\frac{\partial \dot{Q}(x,t)}{\partial x}}$$

$$\therefore \frac{\partial^2 T(x,t)}{\partial x^2} = \cancel{-}\frac{1}{K\cancel{A}}\left[\cancel{-}C_v \cancel{A}\frac{\partial T(x,t)}{\partial t}\right]$$

$$= \frac{C_v}{K}\frac{\partial T(x,t)}{\partial t}$$

we've already seen that $\frac{C_v}{K} = \frac{1}{\alpha}$ where α is the thermal diffusivity.

$$\boxed{\therefore \frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}}$$

1-D heat eq'n. Eq. (7) in lab manual

At this point, one could return to the lab manual to complete a rigorous derivation of the required result. However, we will keep going and present an alternative derivation that will lead to a similar final result.

To make progress, we assume a harmonic time dependence for $T(x,t)$

$$T(x,t) = \underbrace{T_0(x)}_{\substack{\text{position-dependent} \\ \text{amplitude of the temp. oscillation}}} e^{j\omega t}$$

This assumption allows us to evaluate the time derivative

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[T_0(x) e^{j\omega t}\right] = j\omega T_0(x) e^{j\omega t}$$

$\therefore$ the heat eq'n becomes

$$\frac{d^2 T_0(x)}{dx^2} \cancel{e^{j\omega t}} = \frac{j\omega}{\alpha} T_0(x) \cancel{e^{j\omega t}}$$

Eliminate all time dependence. Know focus solely on finding $T_0(x)$.

$$\frac{d^2 T_0(x)}{dx^2} = \frac{jw}{\alpha} T_0(x) \quad (3)$$

Require a fcn that returns itself after taking two derivatives. Try $T_0(x) = T_+ e^{kx} + T_- e^{-kx}$

$$\frac{dT_0(x)}{dx} = k\left(T_+ e^{kx} - T_- e^{-kx}\right)$$

$$\frac{d^2 T_0(x)}{dx^2} = k^2\left(T_+ e^{kx} + T_- e^{-kx}\right)$$

$$= k^2 T_0(x)$$

$\therefore$ (3) becomes:

$$k^2 T_0(x) = \frac{jw}{\alpha} T_0(x)$$

$$\Downarrow$$

$$k = \sqrt{\frac{jw}{\alpha}}$$

How to deal with $\sqrt{j}$?

Recall Euler's Eq'n:

$$e^{j\phi} = \cos\phi + j\sin\phi$$

sub in $\phi = \frac{\pi}{2}$ & use $\cos\pi/2 = 0$, $\sin\pi/2 = 1$

$$e^{j\pi/2} = j$$

$$\therefore \sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4}$$

$$= \cos\pi/4 + j\sin\pi/4$$

$$= \frac{1}{\sqrt{2}}(1+j)$$

$$\boxed{\therefore k = \sqrt{j}\sqrt{\frac{\omega}{\alpha}} = (1+j)\sqrt{\frac{\omega}{2\alpha}}}$$

Back to the amplitude

$$T_0(x) = T_+ e^{kx} + T_- e^{-kx}$$

Require that $T_0(x) \to 0$ as $x \to \infty$.

i.e. the temp. osc. vanishes infinitely far from the heater. This is only possible if $T_+ \to 0$ since e^{kx} diverges for $k > 0$ when $x \to \infty$.

$$\therefore T_0(x) = T_- e^{-kx} = T_- e^{-(1+j)\sqrt{\frac{\omega}{2\alpha}}x}$$

$$= T_- e^{-\sqrt{\frac{\omega}{2\alpha}}x} e^{-j\sqrt{\frac{\omega}{2\alpha}}x}$$

Finally, return to $T(x,t) = T_0(x)e^{j\omega t}$

$$= T_- e^{-\sqrt{\frac{\omega}{2\alpha}}x} e^{-j\sqrt{\frac{\omega}{2\alpha}}x} e^{j\omega t}$$

or

$$\boxed{T(x,t) = T_- \underbrace{e^{-\sqrt{\frac{\omega}{2\alpha}}x}} e^{j(\omega t \underbrace{-\sqrt{\frac{\omega}{2\alpha}}x})}} \quad (4)$$

position-dependent amplitude. Osc. at high freq. (large ω) are attenuated more strongly. Expected low-pass filter behaviour.

↳ position-dependent phase shift of the temp. oscillations. This phase shift was also expected from low-pass filter.

In the termal waves experiment, we focus only on the amplitude:

$$T_0(x) = T_- e^{-\sqrt{\frac{\omega}{2\alpha}}\,x}$$

$$\ln T_0 = \ln T_- - \sqrt{\frac{\omega}{2\alpha}}\, x$$

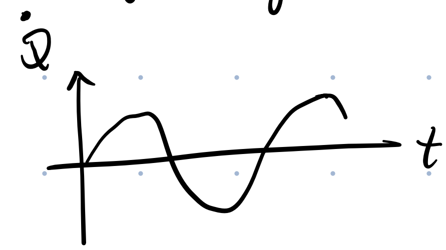
Plot of $\ln T_0$ vs x is a straight line w/ slope

$$m = -\sqrt{\frac{\omega}{2\alpha}}$$

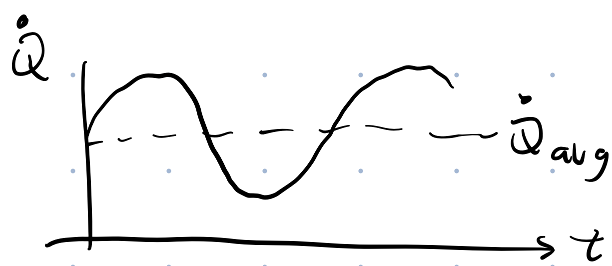
$m \pm \Delta m$ can be used to determine an experimental value for $\alpha \pm \Delta\alpha$.

One could justifiably point out some limitations of the above analysis (after obtaining the 1-D heat eq'n).

- We assumed a sinusoidal/harmonic input power to our heater which triggers a temp. oscillation. However, we can't inject negative power into our heater.



We could only have an oscillation about some average input power.



In such a scenario, there would be a temp. osc. on top of some avg. temp. gradient along the length of the copper rod.

Return to 1-D heat equation

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

If we put in a constant power into our htr, after a sufficiently long time, the Cu bar will reach a equilibrium temp. dist'n s.t. $\frac{\partial T}{\partial t} = 0$

$$\therefore \quad \frac{d^2 T_{avg}}{dx^2} = 0$$

$$\Downarrow$$

$$\frac{dT_{avg}}{dx} = C_1$$

$$\Downarrow$$

$$T_{avg} = C_1 x + C_2$$

If the rod has temp T_0 at $x=0$ and T_L @ $x=L$ (where L is the length of the rod)

Then : $T_{avg}(0) = \boxed{T_0 = c_2}$

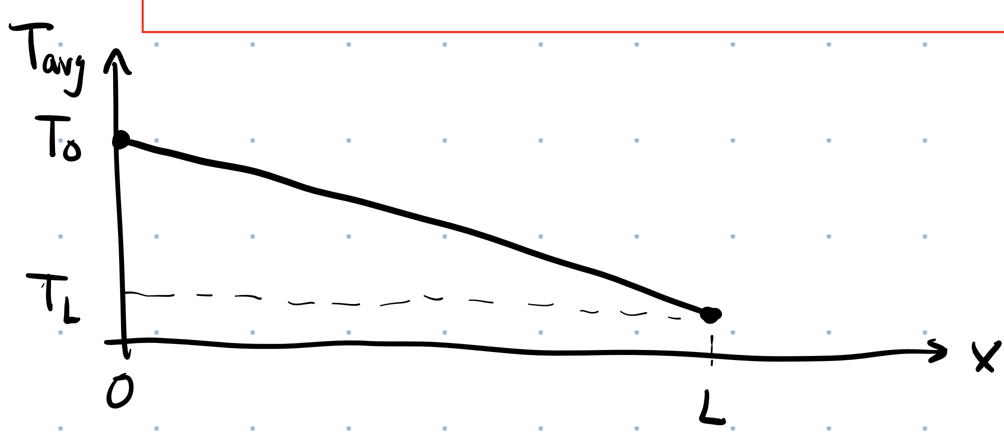
$$\therefore T_{avg}(x) = c_1 x + T_0$$

$$T_{avg}(L) = T_L = c_1 L + T_0$$

$$\therefore c_1 = \frac{T_L - T_0}{L}$$

$$\therefore T_{avg}(x) = \left(\frac{T_L - T_0}{L}\right) x + T_0$$

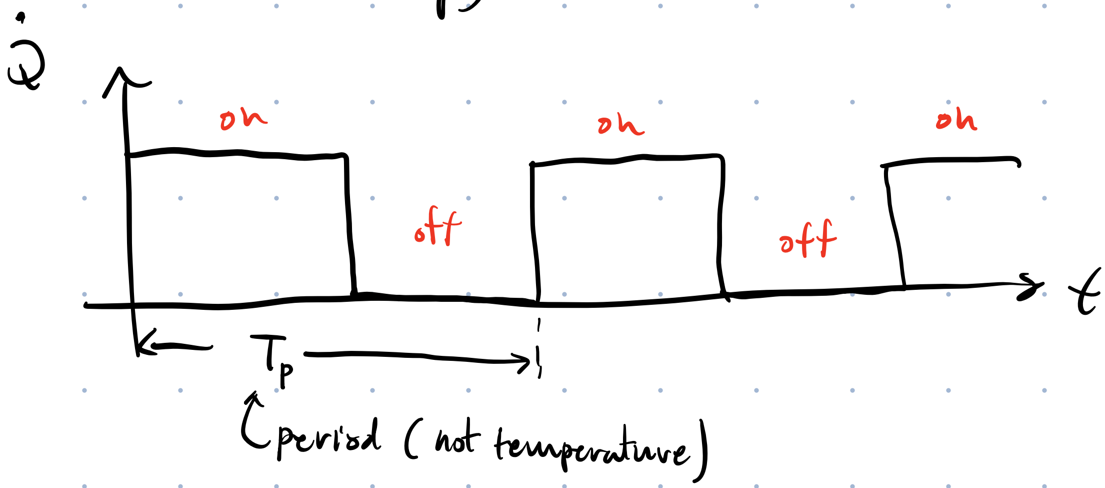
or $$\boxed{T_{avg}(x) = T_0\left(1 - \frac{x}{L}\right) + T_L \frac{x}{L}}$$



The rod's avg. temp. varies linearly along its length.
The end @ $x=0$ has the heater and is hot.
The end @ $x=L$ stays close to the room temp. for a long rod.

If we ignore this linear change in the avg. temp and focus only on the oscillation amplitude about T_{avg}, we can safely apply Eq. (4).

■ A second limitation is that we assumed a sinusoidal input to the heater whereas, in the actual experiment, we apply a square wave (really we turn the power to the htr off and on periodically)



We know that square waves are made up of a sum of sine waves at frequencies of $\frac{1}{T}, \frac{3}{T}, \frac{5}{T}, \dots, \frac{n}{T}$ (all the odd harmonics).

In the thermal waves experiment, we use a Fourier analysis to extract the amplitude of the fundamental harmonic ($n=1$). In such a case, Eq. 4 can be applied. In fact, we could apply Eq. 4 to any of the harmonics since they all represent pure sines.

Furthermore, this analysis would work for any periodic power applied to the heater. The Fourier series tells us that periodic fcns can always be written in terms of a sum of pure sines and cosines.

The n=1 analysis is the easiest because it is the strongest peak $\{$, since $T_0(x) \sim e^{-\sqrt{\frac{\omega}{2\alpha}}x}$, the lowest frequency n=1 peak decays the slowest as x is increased.